J. Intell. Comput. Health Inform. Vol. 4, No. 2, 2023
ISSN: 2715-6923 e-ISSN: 2721-9186, pp.48~53 | DOI: 10.26714/jichi.v4i2.13201.

An Enhanced IS-LM Business Cycle Model for Increasing Income in a Dynamic Economy

Arista Fitri Diana*

Department of Aktuarial Science, Institut Teknologi Statistika dan Bisnis Muhammadiyah Semarang, Jl. Prof Dr.Hamka, KM 1, Ngaliyan Semarang, 50181, Indonesia
*Corresponding author: arista.fitri@itesa.ac.id

Shafira Meiria Rahmasari

Department of Mathematics Education, Sekolah Tinggi Agama Islam Al-Bahjah Cirebon, Jl. Pangeran Cakrabuana Blok Gudang Air, Sendang, Sumber, Kab. Cirebon, 45611, Indonesia

Dhimas Mahardika

Department of Science and Technology, Universitas Nasional Karangturi, Jl. Raden Patah No 182-192 Rejomulyo, Semarang Timur, Kota Semarang, 50127, Indonesia

Article histroy :
Received: 09 JUL 2023
Accepted: 21 SEP 2023
Avalaible online: 30 SEP 2023

Research article

Abstract: This paper introduces an enhanced IS-LM business cycle model by integrating control parameters using the Pontiyagin Maximum Principle Method, aiming to maximize income within economic cycles. It develops a dynamic model incorporating import and consumption rates as controls, showcasing their impact on economic variables through simulations and analytical methodologies. The results exhibit a significant increase in income by up to 10% through the reduction of interest rates and capital stock. The efficiency of the proposed controls is visually demonstrated, providing a robust validation of the methodology used, aligning with prior research, and offering substantial insights into dynamic business cycle modelling for economic analysis and policy-making.

Keywords: IS-LM BUSINESS CYCLE MODEL; DYNAMIC MODEL, ROUTH HURWITZ, MAXIMUM PONTRYAGIN, OPTIMAL CONTROL

1. Introduction

Dynamic systems can be developed in various fields of life, such as engineering, biology, economics, and other social sciences (Radosavljevic et al., 2023). One of the models that is a dynamic system in economics, especially macroeconomics, is the IS-LM business cycle model (Ketut and Tastrawati, 2012). This business cycle model is needed to determine the financial conditions in an economic cycle. Furthermore, this business cycle model can be used as a consideration in policy making in economic institutions. In the economy, especially in the business world, income is one of the indicators that wants to be achieved and even maximize results, so the analysis of a model in the economic cycle is needed to make the right strategy to obtain the optimal income (Maulidina, 2017).

The purpose of this research is to develop a dynamic model of the business cycle using controls based on dynamic models developed in the fields of finance and business. Subsequently, control parameters were added to the model using the Pontiyagin Maximum Principle Method (Séguret, 2024). With this method, you can obtain the best solution to maximize income.

The Business Cycle Model IS-LM with controls was developed by Kaligore, Gabrisch, and Lorenz from the basic model (Gabrisch, 1987). The IS-LM business cycle model is a macroeconomic model that consists of income, capital, and interest rate variables and includes investment, saving, and demand functions. A dynamic model of business cycle is developed by several researchers (Musyaffafi et al., 2018), (Rosmely, Nugrahani, 2016), and (Hidayati et al., 2019). This research is developed because dynamic models can describe the relationship between variables and become solutions to non-dynamic cycle modeling limitations. Model by (Musyaffafi et al., 2018) model IS-LM business cycle model using linear investment function, savings function, and money demand function. Subsequently, the model was further developed by (Rosmely, Nugrahani, 2016), who included an IS-LM business cycle model, an investment function and a nonlinear demand function, and an IS-LM business cycle dynamic model based on a nonlinear function that describes the turnover. Money follows a model that changes continuously. When developing the IS-LM business cycle model, many researchers add external factors influencing the model's dynamics, such as the researchers (Hidayati et al., 2019) adding controls to the IS-LM business cycle model. Adding controls to the research (Hidayati et al., 2019) can reduce delay times so that model rotation cycles can achieve stability faster. From the three above-mentioned researchers, we developed a dynamic model for control of the IS-LM business cycle.

Based on the latest research, we propose dynamic models by adding two controls, namely, import and consumption rates, to achieve the optimal solutions that can maximize income.

In this article, we will be discussed in several stages, that are research methodology, results and discussions, and finally the conclusion.

2. Research Methodology

The research methodology begins with formulating a model, starting with the dynamic IS-LM business cycle model (Chasnov, 2012), which delineates the interplay among income variables, capital stock, and interest rates (Hidayati et al., 2019) by incorporating the investment, savings, and demand for money functions. This model was further adapted into a non-linear IS-LM business cycle model by (Rosmely, Nugrahani, 2016). Their modification involves substituting non-linear forms into the investment, savings, and demand for money functions. (De Cesare et al, 2005) propose the equations for the investment (I), savings (S), and liquidity (L) functions in Eq. (1).

$$I = A\frac{Y^a}{R^b}$$

$$S = sY^aR^b \tag{1}$$

$$L = L_1 + L_2 = gY + \frac{h}{R-r}$$

So by substituting the function in Eq. (1) in the dynamic model of the IS-LM business cycle, a non-linear dynamic model is obtained with the variables income (Y), interest rate (R), and stock of capital (K) as follows in Eq. (2).

$$\frac{dY}{dt} = \alpha\left[A\frac{Y^a}{R^b} + \beta_2 K - sY^aR^b\right]$$

$$\frac{dR}{dt} = \beta_1\left[gY + \frac{h}{R-r} - M\right] \tag{2}$$

$$\frac{dK}{dt} = A\frac{Y^a}{R^b} - (\delta - \beta_2)K$$

Table 1. Parameters description

NOTATIONS	INTERPRETATIONS
$\alpha > 0$	Acceleration due to excess or underinvestment.
$\beta_1 > 0$	Acceleration caused by a shortage or excess demand for money.
$-1 < \beta_2 < 0$	The rate of decline in investment in the capital stock.
$0 < s_1 < 1$	Growth rate of savings on income.
$g > 0$	Quantity demanded of money to income.
$a > 0$	Coefficient of adjustment in the goods market.
$b > 0$	Coefficient of adjustment in the money market.
$A > 0$	Technology productivity.
$M > 0$	Constant money supply.
$h > 0$	The amount of money demanded relative to the interest rate.
$r > 0$	Lowest fixed rate of interest rate.
$\delta > 0$	Capital depreciation constant.

2.1 Equilibrium point of the dynamic model

The equilibrium point is defined as a condition in which no changes are observed in each variable over a given period of time, defined in Eq. (3).

$$\frac{dY}{dt} = \alpha\left[A\frac{Y^a}{R^b} + \beta_1 K - sY^aR^b\right] = 0$$

$$\frac{dR}{dt} = \beta_2\left[AgY + \frac{h}{R-r} - M\right] = 0 \tag{3}$$

$$\frac{dK}{dt} = A\frac{Y^a}{R^b} - (\delta - \beta_2)K = 0$$

We obtained the equilibrium point defined in Eq. (4).

$$E_1 = \left(0, \frac{h+Mr}{M}, 0\right) \tag{4}$$

$$E_2 = \left(\frac{1}{g}\left(M - \frac{h}{R-r}\right), \left(M - \frac{A\delta}{s_1(\delta-\beta_2)}\right)^{\frac{1}{2b}}, \frac{s_1}{\delta}Y^aR^b\right)$$

2.2 Existence of equilibrium point

In this section, we will establish the existence condition of the equilibrium states. The equilibrium points of model (2) are as follows :

$$E_1 = \left(0, \frac{h+Mr}{M}, 0\right) \quad \text{do always exist}$$

$$E_2 = \left(\frac{1}{g}\left(M - \frac{h}{R-r}\right), \left(M - \frac{A\delta}{s_1(\delta-\beta_2)}\right)^{\frac{1}{2b}}, \frac{s_1}{\delta}Y^aR^b\right)$$

The existence condition for E^2 if, $h < M\left(M - \frac{A\delta}{s_1(\delta-\beta_2)}\right)^{\frac{1}{2b}} - r$ with $M > \frac{A\delta}{s_1(\delta-\beta_2)}$

2.3 Local stability analysis

In this section, we will analyze the local stability of equilibrium points. We use eigenvalue and Routh Hurwitz method to analyze the equilibrium points.

First step, we do the linearization of system (2).

$$\frac{d}{dt}\begin{bmatrix}Y\\R\\K\end{bmatrix} = \begin{bmatrix}\frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial R} & \frac{\partial f_1}{\partial K}\\ \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial R} & \frac{\partial f_2}{\partial K}\\ \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial R} & \frac{\partial f_3}{\partial K}\end{bmatrix}(Y,R,K)\begin{bmatrix}\bar{Y}\\ \bar{R}\\ \bar{K}\end{bmatrix}$$

Then with maple application we obtain :

$$\frac{d}{dt}\begin{bmatrix}Y\\R\\K\end{bmatrix} = \begin{bmatrix}\alpha\left[A\frac{Y^a a}{YR^b} - \frac{sY^a aR^b}{Y}\right] & \alpha\left[-A\frac{Y^a b}{rR^b} - \frac{sY^a R^b b}{R}\right] & \alpha\beta_1\\ \beta_2 g & -\frac{\beta_2 h}{(R-r)^2} & 0\\ \frac{AY^a a}{YR^b} & -\frac{AY^a b}{rR^b} & -\delta+\beta\end{bmatrix}\begin{bmatrix}\bar{Y}\\ \bar{R}\\ \bar{K}\end{bmatrix}$$

For a fixed point, we obtain the eigenvalues :

$$\lambda_1 = 0, \lambda_2 = -\frac{M^2\beta}{h}, \lambda_3 = -\delta + \beta_1$$

Because $\lambda_1 = 0$, the stability of fixed points isn't isolated for fixed points

$$E_2 = \left(\frac{1}{g}\left(M - \frac{h}{R-r}\right), \left(M - \frac{A\delta}{s_1(\delta-\beta_2)}\right)^{\frac{1}{2b}}, \frac{s_1}{\delta}Y^a R^b\right)$$

E_1

We substitute E_2 to Jacobian matrix and we get the characteristic polynomial as follows :

With,

$$p_2 = -\alpha\left(\frac{AY^{a-1}a}{R^b}\right) + \beta_1\left(\frac{h}{(R-r)^2}\right)$$
$$p_1 = \alpha\beta_1\left(\left(\frac{AY^{a-1}a}{R^b}\right)\left(-\frac{h}{(R-r)^2}\right) + g\left(\frac{AY^a b}{R^{b+1}}\right)\right) + q_0 + r_0)$$
$$P_0 = -\alpha\beta_1\beta_2 g\left(\frac{AY^a b}{R^{b+1}}\right)$$
$$q_1 = -\alpha\beta_2\left(\frac{AY^{a-1}a}{R^b}\right)$$
$$q_0 = \alpha\beta_1\beta_2\left(-\frac{h}{(R-r)^2}\right)\left(\frac{AY^{a-1}a}{R^b}\right)$$
$$r_2 = (\delta-\beta_2)$$
$$r_1 = -(\delta-\beta_2)\alpha\left(\frac{AY^{a-1}a}{R^b}\right) + \beta_1\left(\frac{h}{(R-r)^2}\right)$$
$$r_0 = (\delta-\beta_2)\alpha\beta_1\beta_2 g\left(\frac{AY^a b}{R^{b+1}}\right)$$

Based on the Routh-Hurwitz criteria for the third degree characteristic equation, the equilibrium point E_2 is locally asymptotically stable if :

$$p_2 + r_2 > 0$$
$$p_0 + q_0 + r_0 > 0$$
$$(p_2 + r_2)(p_1 + q_1 + r_1) > (p_0 + q_0 + r_0)$$

2.4 Analyze of optimal control

Model (2) was modified by reducing import rates and consumption rates to increase income. Import rate control (u_1) is given to the interest rest variable, while the consumption rate control (u_2) is given to the capital stock variable. From the two controls provided, the dynamic system equation with control is obtained:

$$\frac{dY}{dt} = \alpha\left[A\frac{Y^a}{R^b} + \beta_2 K - sY^a R^b\right]$$
$$\frac{dR}{dt} = \beta_2\left[AgY + \frac{h}{R-r} - M\right] - u_1 R \quad (5)$$
$$\frac{dK}{dt} = A\frac{Y^a}{R^b} - (\delta-\beta_2)K - u_2 K$$

with the initial condition, $S(0) > 0, I(0) > 0, H_1(0) > 0, H_2(0) > 0$

Functional objective J formulates optimization problems to identify effective strategies. The optimal control strategy has the aim of controlling interest, rest, and capital stock to maximize income. The objective functional is defined as Eq. (6).

$$J(U) = \int_0^{tf}\left(\omega_1 R(t) + \omega_2 K(t) + \frac{v_1}{2}u_1^2(t) + \frac{v_2}{2}u_2^2(t)\right)dt \quad (6)$$

Where tf is the final time and the coefficients of $\omega_1, \omega_2, v_1, v_2$ Balance cost factor caused by the scale and importance of the four parts of the objective function. To find the optimal control to use,

$$J(u_1^*, u_2^*) = \min\{J(u_1, u_2) \mid u_1, u_2 \in U\} \quad (7)$$

The Hamiltonian function of (5) is :

$$\begin{aligned} H = {} & \omega_1 R + \omega_2 K + \omega_3 H_2 + \frac{v_1}{2}u_1^2 + \frac{v_2}{2}u_2^2 \\ & + \lambda_1\left(\alpha\left[A\frac{Y^a}{R^b} + \beta_2 K - sY^a R^b\right]\right) \\ & + \lambda_2\left(\beta_2\left[AgY + \frac{h}{R-r} - M\right] - u_1 R\right) \\ & + \lambda_4\left(A\frac{Y^a}{R^b} - (\delta-\beta_2)K - u_2\right) \end{aligned} \quad (8)$$

Theorem 2.1 (Dewi, Hana Mutia, 2022) The optimal control for u_1^*, u_2^* and the solution Y^*, R^*, K^* of system (5) that minimize $J(u_1^*, u_2^*)$ on $U = \{u_1, u_2\}$, when there are adjoint variables $\lambda_1, \lambda_2, \lambda_3$ that satisfy :

$$-\frac{d\lambda_1}{dt} = -\lambda_1\alpha\left(A\frac{Y^a b}{YR^b} - \frac{sY^a aR^b}{Y}\right) - \lambda_2\beta_2 g - \lambda_3\left(A\frac{Y^a a}{YR^b}\right)$$

$$-\frac{d\lambda_2}{dt} = -\omega_1 - \lambda_1\alpha\left(A\frac{Y^a a}{RR^b} - \frac{sY^a R^b b}{R}\right) - \lambda_2\left(\beta_2\left(-\frac{h}{(R-r)^2}\right.\right. + \lambda_1\alpha\left(-A\frac{Y^a ab}{YRR^b} - \frac{sY^a Ra^b b}{YR}\right) - \left.\frac{\lambda_3 AY^a ab}{YRR^b}\right) - u_1\Big) + \frac{\lambda_3 AY^a b}{RR^b}$$

$$-\frac{d\lambda_3}{dt} = u_2\lambda_3 - \omega_2$$

Where $\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t) = 0$ are trenversality conditions and the optimal control satisfy the optimality conditions.

Proof.

We use the Pontryagin Maximum Principle to get the optimal control solutions. Differentiate the Hamiltonian equation (8) to u_1, u_2 and evaluate the optimal control variable as :

$$0 = \frac{\partial H}{\partial u_1} = -\lambda_2 R + v_1 u_1$$

$$0 = \frac{\partial H}{\partial u_2} = -\lambda_3 + v_2 u_2$$

From this, we get the optimal control u_1^*, u_2^* as :

$$u_1^* = \frac{\lambda_2 R}{v_1}$$

$$u_2^* = \frac{\lambda_3}{v_2}$$

There for optimal control variables to u_1^*, u_2^* characterize by:

$$u_1^* = \begin{cases} 0 & jika \quad \psi_1^* \le 0 \\ \psi_1^* & jika \quad 0 \le \psi_1^* \le 1 \\ 1 & jika \quad \psi_1^* \ge 1 \end{cases}$$

$$u_2^* = \begin{cases} 0 & jika \quad \psi_2^* \le 0 \\ \psi_2^* & jika \quad 0 \le \psi_2^* \le 1 \\ 1 & jika \quad \psi_2^* \ge 1 \end{cases}$$

where

$$\psi_1^* = \frac{\lambda_2 R}{v_1}$$

$$\psi_2^* = \frac{\lambda_3}{v_2}$$

3. Results and Discussion

Numerical results for both the controlled and uncontrolled models are displayed in this section. Simulation of Eq (2) and (5) was conducted utilizing Maple software version 15, Matlab 2018, and data sourced from (Hidayati et al., 2019). The specific parameter values employed are detailed in Table 2.

Table 2. Parameter values

NOTATION	VALUE	UNIT	REFERENCES
α	1	$time^{-1}$	(Hidayati et al., 2019)
β_1	1	$time^{-1}$	(Hidayati et al., 2019)
β_2	-0.1	$time^{-1}$	(Hidayati et al., 2019)
s_1	0.08	$time^{-1}$	(Hidayati et al., 2019)
g	0.05	$time^{-1}$	(Hidayati et al., 2019)
a	1.03	-	(Hidayati et al., 2019)
b	0.6	-	(Hidayati et al., 2019)
A	0.1	unit	(Hidayati et al., 2019)
M	0.1	unit	(Hidayati et al., 2019)
h	0.01	$time^{-1}$	(Hidayati et al., 2019)
r	0.001	$time^{-1}$	(Hidayati et al., 2019)
δ	0.1	$time^{-1}$	(Hidayati et al., 2019)

Numerical simulation results were derived from the parameters listed in Table 2. Specifically, the model incorporating control will employ the Runge-Kutta Order-4 algorithm. This involves utilizing the forward Runge-Kutta algorithm to solve the system state and the backward Runge-Kutta to complete the co-state system.

3.1 Interest rest with and without control

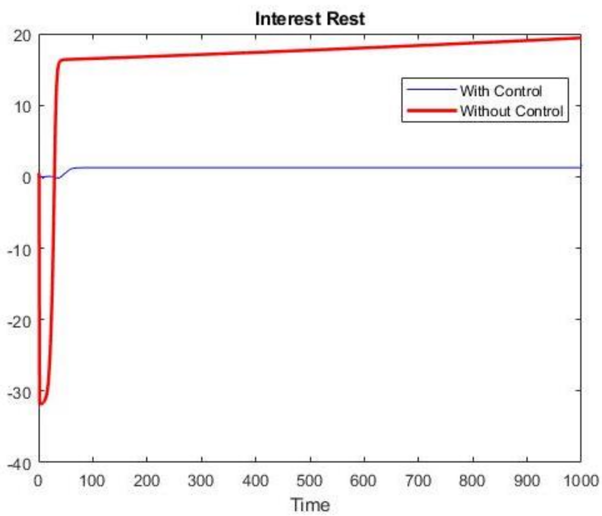


Fig 1. Graph of Interest Rest when given control and not given control

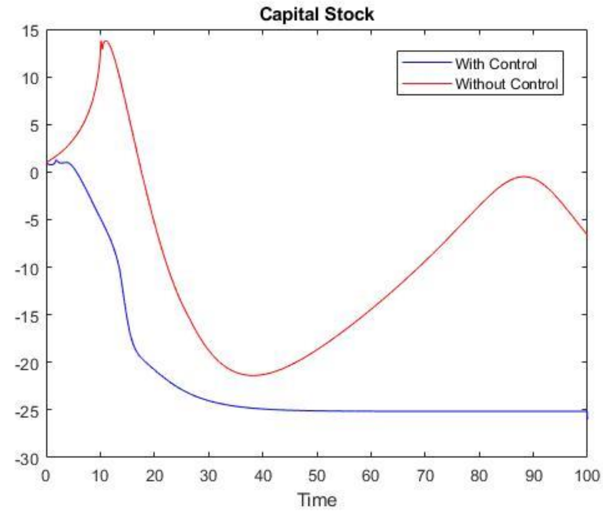


Fig 2. Graph of capital stock when given control and not given control

The simulation model depicted in Fig. 1 demonstrates the impact of implementing import control (u_1) on Interest Rest. The graph illustrates a notable decrease in the Interest Rest following the application of the control, indicating a reduction from its initial value of 19.42 to 1.707.

3.2 Capital stock with and without control

The simulation model depicted in Fig. 2 demonstrates the impact of public consumption (u_2) on the capital stock. The graph illustrates that upon implementing the control, there was a noticeable decrease in the capital stock, declining from -6.6 to -25.15, indicating a reduction in the capital stock following the intervention.

3.3 Income with and without control

The simulation model depicted in Fig. 3 demonstrates that the income has risen by up to 10% subsequent to reducing both interest rates and capital stock.

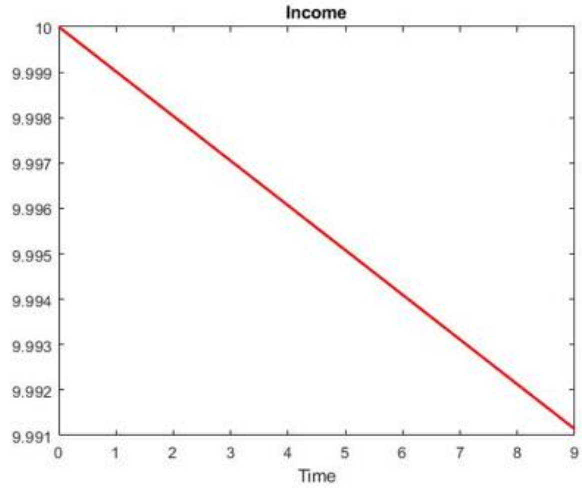
3.4 Control efficiency graph u_1

The controller u_1 in Fig. 4 has a value that ranges from 0 to 1. The percentage of the import rate allocated to interest initially remains stable at point 1 from t = 0 to t = 2.2, after which it gradually decreases until reaching 0 at t = 9. Despite u1 not reaching its maximum of 100%, it still positively influenced increasing income and decreasing interest payments.

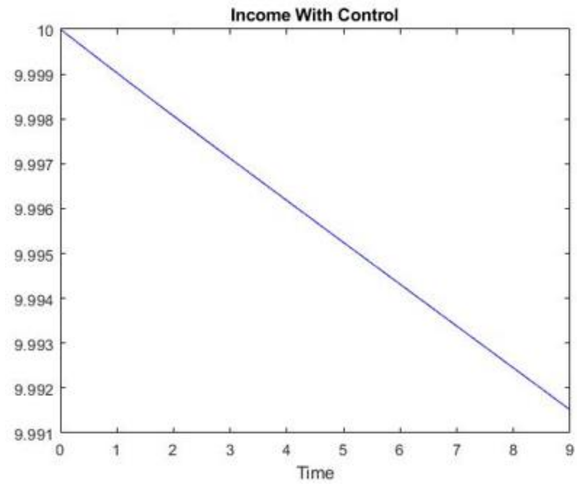
3.5 Control efficiency graph u_2

The controller u_2 in Fig. 5 has a value that ranges from 0 to 1. Public consumption rate was maximized and sustained until t = 1.75, after which it gradually decreased until reaching 0 at t = 9. Utilizing control u_2 below its maximum 100% value still influences an increase in income compartment and a decrease in capital stock compartment.

Based on the results that we have simulated, it proves the efficiency of the method that we have proposed. This research has also been confirmed by (Hidayati et al., 2019) that the application of Pontryagin's Maximum Principle is efficient with the data used. So that the results of this research have strong validation and become the main source which can be used by other researchers in the same research area.



a)



b)

Fig 3. a) Graph of income without control; b) Graph of income when given control

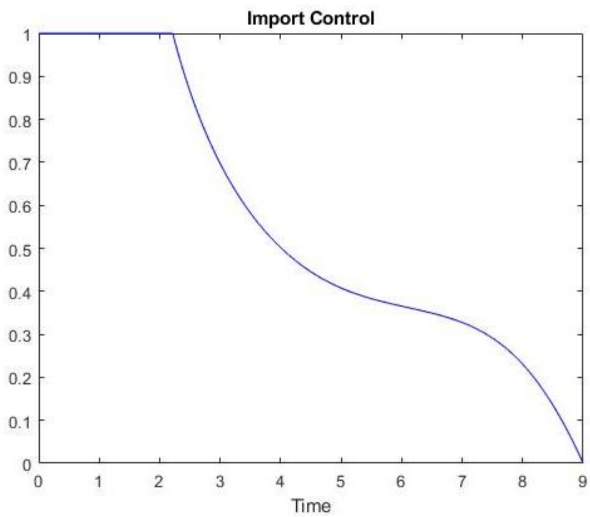


Fig 4. Graph of control condition u_1

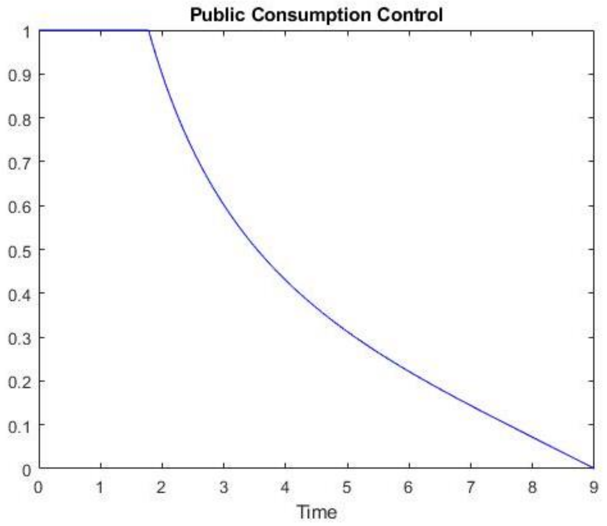


Fig 5. Graph of control condition u_2

4. Conclusion

In this paper, we endeavored to enhance the IS-LM business cycle model by incorporating control parameters utilizing the Pontiyagin Maximum Principle Method. The development of this dynamic model with controls, namely, import and consumption rates, aimed at maximizing income within the economic cycle.

The research methodology involved formulating the dynamic IS-LM business cycle model, transitioning it into a non-linear model, establishing equilibrium points, analyzing local stability, and finally introducing optimal control strategies. Through numerical simulations employing various software tools and referencing pertinent data, the impact of controls on interest rates, capital stock, and overall income was explored and visualized.

Our findings suggest that implementing import and consumption controls can significantly influence economic variables. The simulation outcomes demonstrated a decrease in interest rates and capital stock, ultimately leading to a remarkable increase in income by up to 10%. The control efficiency graphs displayed the dynamic nature of these controls and their gradual impact over time.

This study's robust validation, supported by the efficiency of the Pontryagin's Maximum Principle method, aligns with earlier research (Hidayati et al., 2019), reinforcing the reliability and applicability of our proposed methodology. Consequently, these results provide a strong foundation for further exploration and application within the field of dynamic business cycle modeling.

In summary, this research underscores the potency of incorporating controls within the IS-LM business cycle model to optimize income. The demonstrated efficacy of these controls and the validation of our methodology offer a substantial contribution to the understanding and utilization of dynamic models in economic analysis and policy-making.

Acknowladgements

It is with great appreciation that we acknowledge Mrs. Laelatul Khikmah, M.Si, who shared their insights and experiences in writing scientific articles and provided advice that was invaluable to informing the updated guideline to deploy high-quality JICHI's journal. Specifically, we would like to thank the Faculty of Science and Technology, Institut Teknologi Statistika dan Bisnis Muhammadiyah Semarang, for its support and encouragement throughout the process of conducting this study.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Chasnov, J.R., 2012. Mathematical Biology Lecture notes for MATH 4333 4333, 119.

De Cesare L and Sportelli M 2005, Chaos, Solution and Fractals **25** 2333-244.

Dewi, Hana Mutia, 2022. Optimal Control Of Coronavirus Disease Spread Model In Central Java Province. Mater's Program Thesis. Faculty of Science and Mathematics. University of Diponegoro Semarang.

Gabrisch G, Lorenz HW, 1987. Business Cycle Theory A Survey Of Methods And Conceps. Berlin (DE) : Springer-Verlag.

Hidayati, A.N., Apriliani, E., Usadha, I.G.N.R., 2019. Optimal Control of Dynamic IS-LM Bussiness Cycle Model with Two Time Delay. J. Phys. Conf. Ser. 1373.

Ketut, N., Tastrawati, T., 2012. Model Siklus Bisnis Is-Lm Dengan 2, 44–56.

Matematika, P.S., Matematika, J.P., Matematika, F., Ilmu, D.A.N., Alam, P., Yogyakarta, U.N., 2018. Analisis Bifurkasi Pada Model Matematika Siklus Bisnis. J. Kaji. dan Terap. Mat.

Maulidina, F.I., 2017. Analisis dampak defisit anggaran terhadap inflasi, jumlah uang beredar, dan suku bunga di indonesia 16.

Musyaffafi, U.M., Matematika, P.S., Sains, J., Sains, F., Teknologi, D.A.N., Islam, U., Sunan, N., Surabaya, A., 2018. Analisis Model Kestabilan Siklus Bisnis Investment Saving – Liquidity Money (Is-Lm) Dengan Metode Runge-Kutta Orde Lima Dan Extended Runge-Kutta. Skripsi.

Ndendya, J.Z., Leandry, L., Kipingu, A.M., 2023. A next-generation matrix approach using Routh–Hurwitz criterion and quadratic Lyapunov function for modeling animal rabies with infective immigrants. Healthc. Anal. 4.

Radosavljevic, S., Banitz, T., Grimm, V., Johansson, L.G., Lindkvist, E., Schlüter, M., Ylikoski, P., 2023. Dynamical systems modeling for structural understanding of social-ecological systems: A primer. Ecol. Complex. 56.

Rosmely, Nugrahani, S., 2016. Analisis Dinamik pada Model Siklus Bisnis IS-LM.

Séguret, A., 2024. An optimal control problem for the continuity equation arising in smart charging. J. Math. Anal. Appl. 531, 127891.

Xu, C., Yu, Y., Ren, G., Sun, Y., Si, X., 2023. Stability analysis and optimal control of a fractional-order generalized SEIR model for the COVID-19 pandemic. Appl. Math. Comput. 457, 128210.